Available online at www.sciencedirect.com
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ELSEVIER
J. Math. Anal. Appl. 350 (2009) 777–791
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On some generalizations of Lorentz's almost convergence

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Received 15 September 2007

Available online 12 July 2008

Submitted by B. Cascales

Dedicated to Isaac Namioka for his 80th birthday

Abstract

We investigate an extension of the almost convergence of G.G. Lorentz, further weakening the notion of M-almost convergence we defined in [S. Mercourakis, G. Vassiliadis, An extension of Lorentz's almost convergence and applications in Banach spaces, Serdica Math. J. 32 (2006) 71–98] and requiring that the means of a bounded sequence restricted on a subset M of $\mathbb{N}$ converge weakly in $\ell^\infty(M)$. The case when M has density 1 is of special interest and in this case we derive a result in the direction of the Mean Ergodic Theorem (see Theorem 2).

Keywords: Almost convergence; Shift operator; Banach limit; Weak convergence

0. Introduction

The present paper can be considered as a continuation of the investigation, beginning in [9], of some generalizations of the concept of almost convergence of G.G. Lorentz [8].

We recall that a sequence of real numbers $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is said to be Cesaro summable if the sequence of its arithmetic means $\frac{\alpha_1+\alpha_2+\cdots+\alpha_n}{n}$, $n \in \mathbb{N}$, is convergent in $\mathbb{R}$. If the sequence $\frac{\alpha_j+\alpha_{j+1}+\cdots+\alpha_{j+n-1}}{n}$, $n \in \mathbb{N}$, converges uniformly in $j = 1, 2, \ldots$ to some $x \in \mathbb{R}$, then we say that $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is an almost convergent sequence (to the value x). This notion was introduced by G.G. Lorentz in [8].

Let us denote by $\ell^\infty(\mathbb{N})$ the Banach space of all bounded real sequences with the supremum norm and by $T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ the shift operator, defined by $T(\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) = (\alpha_2, \alpha_3, \ldots, \alpha_{n+1}, \ldots)$. When $\alpha \in \ell^\infty(\mathbb{N})$, we set $f_n^\alpha = \frac{\alpha+T\alpha+\cdots+T^{n-1}\alpha}{n}$, $n \geqslant 1$. In [9] we defined and studied an extension of almost convergence. We called a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ M-almost convergent to $x \in \mathbb{R}$ (where M is a nonempty subset of $\mathbb{N}$) when the sequence $\frac{\alpha_j+\alpha_{j+1}+\cdots+\alpha_{j+n-1}}{n}$, $n \geqslant 1$, converges uniformly in $j \in M$ to x. We also provided examples distinguishing the M-almost convergence from Cesaro summability and from Lorentz's almost convergence. Note that it is easy to

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.07.016

see that the sequence $\alpha \in \ell^\infty(\mathbb{N})$ is M-almost convergent to $x \in \mathbb{R}$ iff the sequence of functions $f_n^\alpha = \frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}$, $n \geqslant 1$, converges uniformly on M to the constant function x.

We now briefly discuss our main results. In the preliminary Section 1 we state the necessary definitions and results and fix the notation. In Section 2 we introduce the notion of *weak M-almost convergence*, a further weakening of almost convergence with respect to the M-almost convergence of [9], i.e. we request that the sequence (f_n^α), defined above, restricted on an $M \subseteq \mathbb{N}$ is weakly convergent in the Banach space $\ell^\infty(M)$. This notion lies between the M-almost convergence and the Cesaro summability and we present examples distinguishing it from both, when the density $d(M)$ of M is less than 1. We also define the sublinear functional w_M^+ (Definition 2), starting from the "double limit condition of Banach" (Theorem 1). Using w_M^+ we obtain a characterization of weak M-almost convergence (Proposition 4). The case when $d(M) = 1$ is different and we have (a slightly more general than) the following result (see Theorem 2).

Theorem A. *Let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ and $M \subseteq \mathbb{N}$ with density $d(M) = 1$. If α is weakly M-almost convergent to $x \in \mathbb{R}$, then there is an N subset of M with $d(N) = 1$, such that the sequence α is N-almost convergent to x.*

The above result is related to the *Mean Ergodic Theorem* (in the special case of the shift operator $T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$) which states that: Let X be a Banach space, $F : X \to X$ a continuous linear map satisfying $\|F^k\| \leqslant M < +\infty$ for $k = 1, 2, \ldots$ and $x \in X$. If the sequence $\frac{x + F(x) + \cdots + F^{n-1}(x)}{n}$, $n \geqslant 1$, has a weak cluster point in X, then (and only then) it converges with respect to the norm in X (see [11, pp. 26–27]).

Finally we obtain a characterization of almost convergence, assuming that the sequence $(f_n^\alpha)_{n\in\mathbb{N}}$ is weakly Cauchy in $\ell^\infty(\mathbb{N})$ (see Theorem 3).

1. Preliminaries

Let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$, then we set

$$d^+(\alpha) = \limsup_n \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} \quad \text{and} \quad d^-(\alpha) = \liminf_n \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n}.$$

A sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is called Cesaro summable in $\mathbb{R}$ if and only if the limit $\lim_n \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = d(\alpha)$ $(= d^+(\alpha) = d^-(\alpha))$ exists. When α is the characteristic function of $A \subseteq \mathbb{N}$, we also write $d^+(A)$ and $d^-(A)$ and these values are called upper and lower density of A, respectively. We introduce the notation

$$\mathcal{D} = \{A \subseteq \mathbb{N}:\ d^+(A) = d^-(A)\}$$

(the class of $A \subseteq \mathbb{N}$ having density $d(A) = d^+(A) = d^-(A)$).

Given a nonempty $M \subseteq \mathbb{N}$ we define the sublinear functional on $\ell^\infty(\mathbb{N})$,

$$d_M^+(\alpha) = \sup_{(t_n),(k_n)} J(\alpha, (t_n), (k_n))$$

where for a sequence (t_n) in M and a subsequence (k_n) of $\mathbb{N}$,

$$J(\alpha, (t_n), (k_n)) = \inf_n \frac{\alpha_{t_n} + \alpha_{t_n+1} + \cdots + \alpha_{t_n+k_n-1}}{k_n}$$

(see [9, Definition 2]). It is then proved that: A bounded sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is M-almost convergent if and only if $d_M^-(\alpha) = d_M^+(\alpha)$ [9, Theorem 1].

Throughout this paper, when $\kappa, \lambda \in \mathbb{N}$ we use the interval notation

$$[\kappa, \lambda] = \{n \in \mathbb{N}:\ \kappa \leqslant n \leqslant \lambda\} \quad \text{and} \quad [\kappa, \lambda) = \{n \in \mathbb{N}:\ \kappa \leqslant n < \lambda\}.$$

We also denote $[\kappa, \infty) = \{n \in \mathbb{N}:\ n \geqslant k\}$.

Let Γ be an infinite set; denote by $\beta\Gamma$ the set of ultrafilters on Γ and by $\ell^\infty(\Gamma)$ the Banach space of all bounded real functions on Γ (with supremum norm). When $A \subseteq \Gamma$ let $\overline{A} = \{u \in \beta\Gamma:\ A \in u\}$. Then $\beta\Gamma$, considered as a topological space with basis $\{\overline{A}:\ A \subseteq \Gamma\}$ coincides with the Stone–Čech compactification of the discrete space Γ (see [13, p. 63] and [4, pp. 228–232]). Let $f \in \ell^\infty(\Gamma)$ and $u \in \beta\Gamma$, then it is proved that there is a real number $f(u)$ (the limit of f with respect to the ultrafilter u) such that: $\forall\varepsilon > 0$ the set $\{\gamma \in \Gamma\colon\ |f(u) - f(\gamma)| < \varepsilon\} \in u$ (see [4,13]). In that way we can extend f to a continuous function on $\beta\Gamma$ and identify isometrically the Banach spaces $\ell^\infty(\Gamma)$ and $C(\beta\Gamma)$ (the space of continuous real functions on $\beta\Gamma$). Hence the dual space $\ell^\infty(\Gamma)^*$ of $\ell^\infty(\Gamma)$ can be identified with the Banach space $\mathcal{M}(\beta\Gamma)$ of regular Borel measures on $\beta\Gamma$ (by the theorem of Riesz).

A sequence (x_n) in a Banach space X is said to be weakly convergent, if there is an $x \in X$ such that $\lim_n x^*(x_n) = x^*(x)$ for all x^* in the dual space X^*; (x_n) is said to be weakly Cauchy, iff the sequence of real numbers $(x^*(x_n))$ is convergent in $\mathbb{R}$ for all x^* in X^*. Note that by Lebesgue's Dominated Convergence Theorem, a uniformly bounded sequence $(x_n) \subseteq \ell^\infty(\Gamma) \equiv C(\beta\Gamma)$ is weakly convergent to $x \in \ell^\infty(\Gamma)$ iff $\lim_n x_n(u) = x(u)$ for all $u \in \beta\Gamma$ and also (x_n) is weakly Cauchy iff the limit $\lim_n x_n(u)$ exists (in $\mathbb{R}$) for all $u \in \beta\Gamma$. We will use the notation $x_n \to_w x$ when the sequence (x_n) converges weakly to $x \in X$.

A positive normed linear functional L on $\ell^\infty(\mathbb{N})$ is called Banach limit if $L(\alpha) = L(T\alpha)$, $\forall\alpha \in \ell^\infty(\mathbb{N})$ (i.e. if L is shift-invariant). It is easy to check that the set of Banach limits $\mathcal{BL}$ is a convex and weak-$*$ compact subset of the unit ball of $\mathcal{M}(\beta\mathbb{N})$. G.G. Lorentz has proved that a sequence $\alpha \in \ell^\infty(\mathbb{N})$ is almost convergent to $x \in \mathbb{R}$ if and only if $L(\alpha) = x$ for every $L \in \mathcal{BL}$ (see [8, Theorem 1] and also [9,14]).

2. Weak M-almost convergence

Definition 1. Let M be a nonempty subset of $\mathbb{N}$ and $T : \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ the shift operator. A sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is called weakly M-almost convergent to the value $x \in \mathbb{R}$ when the sequence of functions $f_n^\alpha = \frac{\alpha+T\alpha+\cdots+T^{n-1}\alpha}{n}$, $n \geqslant 1$, restricted on M, that is, the sequence

$$f_n^\alpha/M : M \to \mathbb{R}, \quad n \in \mathbb{N},$$

converges weakly in the Banach space $\ell^\infty(M)$ to the (necessarily) constant sequence $(x, x, \ldots) \in \ell^\infty(M)$ (see Remarks 1(1) and the Introduction).

Remarks 1. (1) If $f_n^\alpha(k) \to x \in \mathbb{R}$ for a $k \in \mathbb{N}$, then one can check that $f_n^\alpha(p) \to x$, $\forall p \in \mathbb{N}$ (equivalently $d(\alpha) = x$); so (f_n^α) converges pointwise to a constant function on $\mathbb{N}$.

(2) A sequence $(\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is weakly M-almost convergent to $x \in \mathbb{R}$ if and only if

$$f_n^\alpha(u) = \frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}(u) \to x \quad \forall u \in \overline{M},$$

$\overline{M} = \{u \in \beta\mathbb{N}\colon\ M \in u\}$ (see the Preliminaries).

(3) A weakly M-almost convergent sequence is obviously Cesaro summable (to the same value). If M is finite, the weak M-almost convergence coincides with the Cesaro summation of a sequence, so the interesting case is when M is an infinite subset of $\mathbb{N}$, and this is what we assume in the sequel, unless stated otherwise.

Let $u, p \in \beta\mathbb{N}$. Then an addition of ultrafilters can be defined:

$$u + p = \{A \subseteq \mathbb{N}\colon\ \{n \in \mathbb{N}\colon\ T^n A \in u\} \in p\}$$

($T^n A = A - n = \{k \in \mathbb{N}\colon\ k + n \in A\}$), see [13, Section 15]. We note that $(\beta\mathbb{N}, +)$ is a left-topological semigroup. In particular when $h \in \mathbb{N}$ we have (denoting also by h the principal ultrafilter $\{A \subseteq \mathbb{N}\colon\ h \in A\}$)

$$u + h = \{A \subseteq \mathbb{N}\colon\ \{n \in \mathbb{N}\colon\ T^n A \in u\} \in h\} = \{A \subseteq \mathbb{N}\colon\ h \in \{n \in \mathbb{N}\colon\ T^n A \in u\}\} = \{A \subseteq \mathbb{N}\colon\ T^h A \in u\}.$$

So $(u + h)(A) = u(T^h A)$ $\forall A \subseteq \mathbb{N}$.

When $\alpha \in \ell^\infty(\mathbb{N})$, one can use the fact that $u(\alpha)$ is the limit of α with respect to the ultrafilter u (see the Preliminaries) and prove that $(u + h)(\alpha) = u(T^h\alpha)$ $\forall\alpha \in \ell^\infty(\mathbb{N})$.

The following result is the analogue of Proposition 1 in [9] holding in the case of M-almost convergence.

Proposition 1. *If the sequence $(\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ is weakly M-almost convergent to the value $x \in \mathbb{R}$ for an $M \subseteq \mathbb{N}$, then it is also weakly $(M - h)$-almost convergent to x, $\forall h \in \mathbb{N}$ ($M - h = \{k \in \mathbb{N}\colon\ k + h \in M\}$).*

Proof. We have that

$$\frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}(u) \to x \quad \forall u \in \overline{M}.$$

Let $h \in \mathbb{N}$ and $u \in \beta\mathbb{N}$. Then $u \in \overline{M-h} \Leftrightarrow M - h \in u \Leftrightarrow T^h M \in u \Leftrightarrow M \in u + h \Leftrightarrow u + h \in \overline{M}$. Hence $\overline{M} = \{v + h\colon v \in \overline{M-h}\}$.

Let $v \in \overline{M-h}$. Then

$$\frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}(v+h) = \frac{\alpha(v+h) + \cdots + T^{n-1}\alpha(v+h)}{n} = \frac{T^h\alpha(v) + \cdots + T^{h+n-1}\alpha(v)}{n}$$

so the limit

$$\lim_n \frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}(v) = \lim_n \frac{T^h\alpha + \cdots + T^{h+n-1}\alpha}{n}(v) = \lim_n \frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}(v+h) = x$$

exists. Since v was an arbitrary element of $\overline{M-h}$, we conclude that α is weakly $(M-h)$-almost convergent to the value x. $\square$

Corollary 1. *Let $M = \{m_1 < m_2 < \cdots < m_n < \cdots\}$ be a syndetic subset of $\mathbb{N}$ (i.e. the differences $m_{n+1} - m_n$ are bounded). If the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is weakly M-almost convergent to $x \in \mathbb{R}$, then it is almost convergent (to the same value).*

Proof. The previous proposition implies that α is weakly $(M-h)$-almost convergent to x $\forall h \in \mathbb{N}$. Since M is syndetic, there is $k \in \mathbb{N}$ such that $\bigcup_{h=0}^{k-1}(M-h) = \mathbb{N}$.

Let $u \in \beta\mathbb{N}$. Then there is $h \in \{0, 1, \ldots, k-1\}$ with $u \in \overline{M-h}$. Hence

$$\frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}(u) \to x \quad \forall u \in \beta\mathbb{N},$$

and therefore

$$\frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n} \to_w (x, x, \ldots)$$

(converges weakly) in $\ell^\infty(\mathbb{N})$. The Mean Ergodic Theorem (see the Introduction) implies that α is almost convergent to x. $\square$

The following lemma was proved in [9] (see [9, Lemma 1]).

Lemma 1. *Let $(\Delta_n)_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into intervals of the positive integers such that:*

1. $\max \Delta_n + 1 = \min \Delta_{n+1}$.
2. *The cardinality* $|\Delta_n| \to_n \infty$.

Set $A = \bigcup_{j=0}^\infty \Delta_{2j+1}$ and let $u \in \beta\mathbb{N} \setminus \mathbb{N}$. Then there is $n_0 \geqslant 0$ such that either $\{n \geqslant 0\colon T^n A \in u\} = [n_0, \infty)$ or $\{n \geqslant 0\colon T^n A \in u\} = [0, n_0)$.

We now give an example of a sequence $\alpha \in \ell^\infty(\mathbb{N})$ which is weakly M-almost convergent for an infinite subset M of $\mathbb{N}$, while α is not N-almost convergent for any infinite subset N of $\mathbb{N}$.

Example 1. Let $M = \bigcup_{n=1}^\infty [n^2, n^2 + n]$ (with $d(M) = \frac{1}{2}$) and let also $A = \bigcup_{\{n \text{ is odd}\}} [n^2 + [\frac{3n}{2}], (n+1)^2)$ and $B = \bigcup_{\{n \text{ is even}\}} [n^2 + [\frac{3n}{2}], (n+1)^2)$ (where $[x]$ denotes the integral part of $x \in \mathbb{R}$). We set $\alpha = \chi_A - \chi_B \in \ell^\infty(\mathbb{N})$ (χ_A is the characteristic function of the set A). One can easily verify that $d(\alpha) = 0$ calculating the limit $\lim_n \frac{2-2+3-3+\cdots+(-1)^{n+1}(2n+1-[\frac{3n}{2}])}{(n+1)^2}$ (since $\lim \frac{(n+1)^2}{n^2} = 1$). If $u \in \beta\mathbb{N} \setminus \mathbb{N}$ we have

$$f_n^\alpha(u) = \frac{\alpha + T\alpha + \cdots + T^{n-1}\alpha}{n}(u) = \frac{u(A) - u(B) + u(TA) - u(TB) + \cdots + u(T^{n-1}A) - u(T^{n-1}B)}{n}$$
$$= \frac{u(A) + u(TA) + \cdots + u(T^{n-1}A)}{n} - \frac{u(B) + u(TB) + \cdots + u(T^{n-1}B)}{n}$$

and each of these two quotients converges to 0 or 1 (see Lemma 1). Consequently $f_n^\alpha(u) \to 0, 1$ or -1 $\forall u \in \beta\mathbb{N} \setminus \mathbb{N}$. When $u \in \overline{M} \setminus M$ we have $u(T^n\alpha) = u(T^nA) - u(T^nB) = 0$ $\forall n \in \mathbb{N}$, since for each $n \in \{0, 1, \dots\}$ the sets $T^nA \setminus F_n^1$ and $T^nB \setminus F_n^2$ are contained in $\mathbb{N} \setminus M$ for some finite sets F_n^1, F_n^2. We conclude that $f_n^\alpha(u) = 0$ $\forall n \in \mathbb{N}$, $\forall u \in \overline{M} \setminus M$ and since $d(\alpha) = 0$, it follows that $f_n^\alpha/M \to_w 0 \in \ell^\infty(M)$.

Let now N be an infinite subset of $\mathbb{N}$. Then one of the following holds:

1. An infinite subsequence (t_n) of N is contained in M. If infinite of these points belong to the set of even blocks of M, then one can see (calculating the means from t_n to the next square) that $d_N^-(\alpha) \leqslant -\frac{1}{4}$, while if infinite of them belong to the set of odd blocks of M we have $d_N^+(\alpha) \geqslant \frac{1}{4}$.
2. An infinite subsequence (t_n) of N is contained in $\mathbb{N} \setminus M$ and not contained in $A \cup B$. If infinite of these points are contained in the set of even blocks of $\mathbb{N} \setminus M$ then similarly we have $d_N^-(\alpha) \leqslant -\frac{1}{2}$, while if infinite of them are contained in the set of odd blocks we have $d_N^+(\alpha) \geqslant \frac{1}{2}$.
3. An infinite subsequence (t_n) of N is contained in A (or in B). If for infinite n the distances from t_n to the next square are unbounded, then we can take the mean from t_n to the end of the corresponding block of A (or B) and conclude that $d_N^+(\alpha) = 1$ ($d_N^-(\alpha) = -1$, respectively). Otherwise, we take the mean from t_n to the square after the next and obtain $d_N^-(\alpha) \leqslant -\frac{1}{4}$ ($d_N^+(\alpha) \geqslant \frac{1}{4}$, respectively).

In any case $d_N^-(\alpha) < d_N^+(\alpha)$ (since $d_N^-(\alpha) \leqslant d(\alpha) \leqslant d_N^+(\alpha)$ from [9, Proposition 3]) and hence α is not N-almost convergent for any infinite $N \subseteq \mathbb{N}$ (see [9, Theorem 1]).

Remark 2. We observe that for any $\delta > 0$ we can similarly construct an $M \subseteq \mathbb{N}$ with $d(M) > 1 - \delta$ and a sequence $\alpha \in \ell^\infty(\mathbb{N})$ such that $f_n^\alpha/M \to_w 0$ while the convergence is not uniform on any infinite subset of $\mathbb{N}$. M will be an infinite disjoint union of intervals and $\alpha = \chi_A - \chi_B$ for suitable $A, B \subseteq \mathbb{N} \setminus M$.

The case when $d(M) = 1$ is different and will be considered later on (see Theorem 2).

Example 2. There is an $A \subseteq \mathbb{N}$ with $d(A) = \frac{1}{2}$ ($A \in \mathcal{D}$), while for $\alpha = \chi_A$ the sequence f_n^α/M does not converge weakly for any infinite $M \subseteq \mathbb{N}$ (see also [9, examples (1)]).

Let $A = \bigcup_{n=1}^\infty [n^2, n^2 + n]$ and M be an infinite subset of $\mathbb{N}$. It is easy to check that $d(A) = \frac{1}{2}$. Lemma 1 implies that $f_n^\alpha(u) \to_n 0$ or 1 $\forall u \in \beta\mathbb{N} \setminus \mathbb{N}$, therefore f_n^α/M (converges pointwise but) does not converge weakly, since the limit function is not continuous on βM.

Consequently, when $\alpha \in \ell^\infty(\mathbb{N})$ the following implications hold:

α is almost convergent $\Rightarrow$ α is M-almost convergent (for any $M \subseteq \mathbb{N}$) $\Rightarrow$ α is weakly M-almost convergent $\Rightarrow$ α is Cesaro summable,

while the previous examples and Example 1 of [9] show that the reverse implications do not hold.

In the sequel we are going to define a sublinear functional on $\ell^\infty(\mathbb{N})$, with the help of which we obtain a characterization of weak M-almost convergence. The idea for defining this functional stems from the "double limit condition of Banach" (see [12] and [13, p. 16]).

Theorem 1 *(Double limit condition of Banach). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^\infty(\Gamma)$ (Γ an infinite set) with supremum norm $\|f_n\|_\infty \leqslant K < +\infty$, $\forall n \in \mathbb{N}$. Then the following are equivalent*:

1. *$f_n \to_w 0$ in $\ell^\infty(\Gamma)$.*
2. *For every sequence $(\gamma_m)_{m\in\mathbb{N}}$ in Γ such that for all $n \in \mathbb{N}$ the sequence $(f_n(\gamma_m))_{m\in\mathbb{N}}$ converges, we have that*

$$\lim_n\Bigl(\lim_m f_n(\gamma_m)\Bigr) = 0.$$

We now give the following definition (cf. the definition of $d_M^+(\alpha)$ in the Preliminaries).

Definition 2. Given $\alpha = (\alpha_n) \in \ell^\infty(\mathbb{N})$, a sequence (γ_m) in M and a sequence $(m(n))$ in $\mathbb{N}$ (not necessarily increasing), we set

$$H\bigl(\alpha, (\gamma_m), \bigl(m(n)\bigr)\bigr) = \limsup_n \inf_{m \geqslant m(n)} \frac{\alpha_{\gamma_m} + \alpha_{\gamma_m+1} + \cdots + \alpha_{\gamma_m+n-1}}{n}.$$

Then we define

$$w_M^+(\alpha) = \sup_{(\gamma_m),(m(n))} H\bigl(\alpha, (\gamma_m), \bigl(m(n)\bigr)\bigr) \quad \text{and} \quad w_M^-(\alpha) = -w_M^+(-\alpha).$$

Remarks 3. (1) One can check that w_M^+ is a bounded sublinear functional on $\ell^\infty(\mathbb{N})$ (see also the expression of Proposition 3), i.e. that

(a) $w_M^+(\lambda\alpha) = \lambda w_M^+(\alpha)$ $\forall \alpha \in \ell^\infty(\mathbb{N})$ and $\forall \lambda \geqslant 0$,
(b) $w_M^+(\alpha+\beta) \leqslant w_M^+(\alpha) + w_M^+(\beta)$ $\forall \alpha, \beta \in \ell^\infty(\mathbb{N})$, and
(c) $w_M^+(\alpha) \leqslant \|\alpha\|_\infty$ $\forall \alpha \in \ell^\infty(\mathbb{N})$.

(2) The sequence (γ_m) considered in M is arbitrary. Essentially the cases we have to examine are:

(a) $\{\gamma_m\colon m \in \mathbb{N}\}$ is a finite set, so we can consider that $\gamma_m = \gamma$, $\forall m \in \mathbb{N}$ (a constant sequence). Then the lim sup we take equals $d^+(\alpha)$ and hence $w_M^+(\alpha) \geqslant d^+(\alpha)$ $\forall \alpha \in \ell^\infty(\mathbb{N})$ (if in particular M is finite, $w_M^+(\alpha) = d^+(\alpha)$).
(b) (γ_m) has a strictly increasing subsequence, so we consider that (γ_m) is itself strictly increasing.

When $A \subseteq \mathbb{N}$ and $\alpha = \chi_A$, $A \subseteq \mathbb{N}$ we denote $w_M^+(A) = w_M^+(\alpha)$ and introduce the notation

$$\mathcal{W}_M = \bigl\{A \subseteq \mathbb{N}\colon w_M^+(A) = w_M^-(A)\bigr\}.$$

We recall that in [9] we have introduced the similar notation $\mathcal{D}_M = \{A \subseteq \mathbb{N}\colon d_M^+(A) = d_M^-(A)\}$, the class of sets $A \subseteq \mathbb{N}$ having M-Banach density $d_M(A) = d_M^+(A) = d_M^-(A)$; in particular $\mathcal{D}_\mathbb{N}$ is the class of subsets A of $\mathbb{N}$ having Banach density. It is clear that for any $M \subseteq \mathbb{N}$, $M \neq \emptyset$ we have (see also [9, Remarks 2])

$$\mathcal{D}_\mathbb{N} \subseteq \mathcal{D}_M \subseteq \mathcal{W}_M \subseteq \mathcal{D}.$$

We know from results of [9] and from Example 2 that there are $M \subseteq \mathbb{N}$ such that $\mathcal{D}_\mathbb{N} \subsetneq \mathcal{D}_M \subsetneq \mathcal{D}$ and $\mathcal{W}_M \subsetneq \mathcal{D}$. We shall show by Example 4 that it can be $\mathcal{D}_M \subsetneq \mathcal{W}_M$ even for an $M \subseteq \mathbb{N}$ of density 1.

Proposition 2. *Let $\alpha \in \ell^\infty(\mathbb{N})$ and $M \subseteq \mathbb{N}$, $M \neq \emptyset$. Then we have*

$$d_M^-(\alpha) \leqslant w_M^-(\alpha) \leqslant d^-(\alpha) \leqslant d^+(\alpha) \leqslant w_M^+(\alpha) \leqslant d_M^+(\alpha).$$

Proof. We shall prove that the relation $w_M^+(\alpha) \leqslant d_M^+(\alpha)$ holds $\forall \alpha \in \ell^\infty(\mathbb{N})$.

Let $\varepsilon > 0$ and pick (γ_m) a sequence in M and $(m(n))$ a sequence in $\mathbb{N}$ such that

$$\limsup_n \inf_{m \geqslant m(n)} \frac{\alpha_{\gamma_m} + \cdots + \alpha_{\gamma_m+n-1}}{n} > w_M^+(\alpha) - \varepsilon.$$

Then, there is a subsequence (k_n) of $\mathbb{N}$ with

$$\inf_{m \geqslant m(k_n)} \frac{\alpha_{\gamma_m} + \cdots + \alpha_{\gamma_m+k_n-1}}{k_n} > w_M^+(\alpha) - \varepsilon \quad \forall n \in \mathbb{N}.$$

In particular

$$\frac{\alpha_{\gamma_{m(k_n)}} + \cdots + \alpha_{\gamma_{m(k_n)}+k_n-1}}{k_n} > w_M^+(\alpha) - \varepsilon \quad \forall n \in \mathbb{N},$$

hence

$$\inf_n \frac{\alpha_{\gamma_{m(k_n)}} + \cdots + \alpha_{\gamma_{m(k_n)}+k_n-1}}{k_n} \geqslant w_M^+(\alpha) - \varepsilon$$

which implies the result. One can immediately obtain the other inequalities (cf. also [9, Proposition 3] and Remarks 3(2)). □

Remark 4. There are $M, A \subseteq \mathbb{N}$ with $w_M^+(A) < d_M^+(A)$.

Let $M = \bigcup_{n=1}^{\infty}[n^3, n^3 + 3n^2 + n + 1]$ and $A = \bigcup_{n=1}^{\infty}[n^3 + 3n^2 + 2n + 1, (n+1)^3)$. One can easily check that $d(A) = 0$, $d_M^+(A) = \frac{1}{2}$ (consider the $2n$-means starting from points $n^3 + 3n^2 + n + 1$) and also that for any strictly increasing sequence (γ_m) in M the relations

$$\frac{\alpha_{\gamma_m} + \cdots + \alpha_{\gamma_m+n-1}}{n} = 0, \quad n \in \mathbb{N},$$

hold (for fixed n and all except for finite m). So we have (see Remarks 3(2)) $w_M^+(A) = 0 < \frac{1}{2} = d_M^+(A)$.

Proposition 3. *Let M be an infinite subset of $\mathbb{N}$. Then*

$$w_M^+(\alpha) = \sup_{u \in \overline{M}} \limsup_n f_n^\alpha(u), \quad \alpha \in \ell^\infty(\mathbb{N}).$$

Proof. Let z be the right-hand side of the equation. We first prove that $w_M^+(\alpha) \leqslant z$. Let $m \in M$. We have (see also Remarks 1(1))

$$(1) \qquad d^+(\alpha) = \limsup_n f_n^\alpha(m) \leqslant z.$$

So we consider a strictly increasing sequence (γ_m) in M (see Remarks 3(2)) and a sequence $(m(n))$ in $\mathbb{N}$. Let

$$(2) \qquad x_n = \inf_{m \geqslant m(n)} f_n^\alpha(\gamma_m), \quad n \in \mathbb{N}.$$

It suffices to show that $\limsup x_n \leqslant z$. Let $u \in \overline{M} \setminus M$ such that the set $\{\gamma_m\colon m \in \mathbb{N}\} \in u$. Suppose there is $n_0 \in \mathbb{N}$ with $f_{n_0}^\alpha(u) < x_{n_0}$ and choose $\delta > 0$ such that $f_{n_0}^\alpha(u) < x_{n_0} - \delta$. By the continuity of $f_{n_0}^\alpha$ on $\overline{M}$ there is an infinite subset B_0 of M, $B_0 \in u$ such that

$$f_{n_0}^\alpha(B_0) \subseteq (-\infty, x_{n_0} - \delta).$$

Then $(\gamma_m) \cap B_0 \in u$ is an infinite set, so there is $m \geqslant m(n_0)$ such that $f_{n_0}^\alpha(\gamma_m) < x_{n_0} - \delta$ and by (2) we have a contradiction. Hence $f_n^\alpha(u) \geqslant x_n$ $\forall n \in \mathbb{N}$, implying $\limsup_n f_n^\alpha(u) \geqslant \limsup_n x_n$ and the result follows from this inequality combined with (1).

We now prove the reverse inequality $z \leqslant w_M^+(\alpha)$. Consider an $m \in M$. Then

$$\limsup_n f_n^\alpha(m) = d^+(\alpha) \leqslant w_M^+(\alpha)$$

(see Remarks 3(2)). Let now $u \in \overline{M} \setminus M$ and $\varepsilon > 0$. The continuity of f_n^α on $\overline{M}$ implies: for $n \in \mathbb{N}$ there is a subset B_n of M, $B_n \in u$ such that

$$(3) \qquad f_n^\alpha(B_n) \subseteq \big(f_n^\alpha(u) - \varepsilon, +\infty\big).$$

Since $B_1 \cap B_2 \cap \cdots \cap B_k \in u$ $\forall k \in \mathbb{N}$, are infinite sets, we can choose a strictly increasing sequence (γ_m) in M with $\gamma_m \in \bigcap_{i=1}^m B_i$, $m \in \mathbb{N}$. By the choice of (γ_m) and (3) we have $f_n^\alpha(\gamma_m) > f_n^\alpha(u) - \varepsilon$, $n \in \mathbb{N}$ (for fixed n and for all except for finite m).

So, for $n \in \mathbb{N}$ there is $m(n) \in \mathbb{N}$ such that $\inf_{m \geqslant m(n)} f_n^\alpha(\gamma_m) \geqslant f_n^\alpha(u) - \varepsilon$. Hence

$$\limsup_n \inf_{m \geqslant m(n)} f_n^\alpha(\gamma_m) \geqslant \limsup_n f_n^\alpha(u) - \varepsilon \quad \text{for any } \varepsilon > 0$$

and so

$$\limsup_n f_n^\alpha(u) \leqslant w_M^+(\alpha) \quad \forall u \in \overline{M} \setminus M. \qquad \square$$

Proposition 4. *Let $\alpha \in \ell^\infty(\mathbb{N})$ and $M \subseteq \mathbb{N}$. The sequence α is weakly M-almost convergent to $x \in \mathbb{R} \Leftrightarrow w_M^+(\alpha) = w_M^-(\alpha) = x$.*

Proof. By Proposition 3, it is easy to see that

$$w_M^-(\alpha) = \inf_{u \in \overline{M}} \liminf_n f_n^\alpha(u), \quad \alpha \in \ell^\infty(\mathbb{N}).$$

The result follows directly, since:

α is weakly M-almost convergent to $x \in \mathbb{R} \Leftrightarrow \lim_n f_n^\alpha(u) = x$ $\forall u \in \overline{M} \Leftrightarrow \liminf_n f_n^\alpha(u) = \limsup_n f_n^\alpha(u) = x$ $\forall u \in \overline{M} \Leftrightarrow w_M^+(\alpha) = w_M^-(\alpha) = x$. $\square$

Let $\alpha \in \ell^\infty(\mathbb{N})$. If there is a subsequence (k_n) of the positive integers such that the sequence $f_{k_n}^\alpha$ converges weakly in $\ell^\infty(\mathbb{N})$, then the Mean Ergodic Theorem (see the Introduction) implies that α is almost convergent (i.e. (f_n^α) converges uniformly on $\mathbb{N}$). We give an example showing that the same is not true for the weak M-almost convergence (see also [9, Example 3]).

Example 3. Let $M = \bigcup_{n=1}^\infty [n^3, n^3 + 3n^2 + 2n + 1]$ with $d(M) = 1$. Consider a subset A of $\mathbb{N}$ with $d^+(A) = 1$ and $d^-(A) = 0$. We construct a sequence $\alpha \in \ell^\infty(\mathbb{N})$ as follows: We set $\alpha(i) = 1$, if $i = n^3 + 3n^2 + 2n + j$, $1 \leqslant j \leqslant n$ and $\chi_A(j) = 1$. Otherwise we set $\alpha(i) = 0$. It is easy to check that $d(\alpha) = 0$.

Consider now (k_n), (l_n) subsequences of the positive integers such that

$$d^+(A) = \lim_n \frac{|A \cap [1, l_n]|}{l_n} = 1$$

and

$$d^-(A) = \lim_n \frac{|A \cap [1, k_n]|}{k_n} = 0.$$

Let (γ_m) be a sequence in M. We distinguish cases:

1. (γ_m) is constant. Then $f_{k_n}^\alpha(\gamma_m) \to_n 0$ (since $d(\alpha) = 0$).
2. (γ_m) is strictly increasing. Let $\varepsilon > 0$ and consider n_0 large enough such that

$$\frac{|A \cap [1, k_n]|}{k_n} < \varepsilon$$

for $n \geqslant n_0$. Fix $n \geqslant n_0$. Then pick $m(n) \in \mathbb{N}$ such that for $m \geqslant m(n)$ the interval $[\gamma_m, \gamma_m + k_n - 1]$ does not intersect the next block of M. So, for $m \geqslant m(n)$ we have that $f_{k_n}^\alpha(\gamma_m) < \varepsilon$. For any subsequence $(\gamma_m{}')$ of (γ_m) for which the limits $\lim_m f_{k_n}^\alpha(\gamma_m{}')$, $n \in \mathbb{N}$, exist, it follows that $\lim_n \lim_m f_{k_n}^\alpha(\gamma_m{}') = 0$.

By the double limit condition (Theorem 1) we conclude that $f_{k_n}^\alpha / M \to_w 0$, while (obviously) f_n^α / M cannot converge weakly (it suffices to consider $f_{l_n}^\alpha(k^3 + 3k^2 + 2k + 1) \simeq 1$ for large enough n and all except for finite $k \in \mathbb{N}$).

Example 4. $\mathcal{D}_M \subsetneq \mathcal{W}_M$ even for a subset M of $\mathbb{N}$ of density 1.

Let $M = \bigcup_{n=1}^{\infty}[n^3, n^3 + 3n^2 + n + 1]$ with $d(M) = 1$. Let also $A = \bigcup_{n=1}^{\infty}[n^3 + 3n^2 + 2n + 1, (n+1)^3)$ with $d(A) = 0$ and let $\alpha = \chi_A$. By Lemma 1 we have $f_n^{\alpha}(u) \to 0$ or 1 $\forall u \in \beta\mathbb{N} \setminus \mathbb{N}$. In particular $f_n^{\alpha}(u) \to 0$ when $M \in u$ (because there are finite sets $F_n \subseteq \mathbb{N}$ with $T^n A \setminus F_n \subseteq \mathbb{N} \setminus M$ $\forall n \in \mathbb{N}$). Since $d(A) = 0$ we also have $f_n^{\alpha}(m) \to_n 0$ $\forall m \in M$. So $f_n^{\alpha}/M \to_w 0$, while it is easy to check that $d_M^+(\alpha) = \frac{1}{2}$, hence the convergence is not uniform on M.

Nevertheless, we will show that if α is weakly M-almost convergent for an $M \subseteq \mathbb{N}$ with $d(M) = 1$, there is a subset N of M with $d(N) = 1$ such that α is N-almost convergent.

Lemma 2. *Let $M \subseteq \mathbb{N}$ with $d(M) = 1$ and $\varepsilon > 0$. Let $b = \chi_M \in \ell^{\infty}(\mathbb{N})$ and A be the set*

$$A = \left\{ m \in M \colon \frac{b_m + b_{m+1} + \cdots + b_{m+k-1}}{k} > 1 - \varepsilon \ \forall k \in \mathbb{N} \right\}.$$

Then $d(A) = 1$.

Proof. Suppose A does not have density 1, or equivalently $d^+(\complement A) > \delta > 0$ (where $\complement A = \mathbb{N} \setminus A$). Consider a subsequence (n_k) of $\mathbb{N}$ such that

$$\frac{|\complement A \cap [1, n_k]|}{n_k} \to d^+(\complement A) > \delta.$$

Since $d(M) = 1$, we get that

$$\frac{|\complement A \cap M \cap [1, n_k]|}{n_k} \to d^+(\complement A).$$

For each $m \in \complement A \cap M$ we pick a $k_m \in \mathbb{N}$ such that

$$\frac{b_m + b_{m+1} + \cdots + b_{m+k_m-1}}{k_m} \leqslant 1 - \varepsilon. \tag{1}$$

Fix an n_k and consider the set $D = \complement A \cap M \cap [1, n_k]$. We set

$$\begin{aligned}
&\alpha_1 = \min D, \\
&\alpha_2 = \min\bigl(D \setminus [\alpha_1, \alpha_1 + k_{\alpha_1} - 1]\bigr), \\
&\vdots \\
&\alpha_l = \min\left(D \setminus \bigcup_{i=1}^{l-1} [\alpha_i, \alpha_i + k_{\alpha_i} - 1] \right)
\end{aligned}$$

is the maximum element of this form in D satisfying $\alpha_l + k_{\alpha_l} - 1 \leqslant n_k$.

We distinguish the following cases:

1. D does not have an element greater than $\alpha_l + k_{\alpha_l} - 1$. Then (from (1) we have)

$$\begin{aligned}
\frac{b_1 + b_2 + \cdots + b_{n_k}}{n_k} &\leqslant (1 - \varepsilon)\frac{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_l}}{n_k} + \frac{n_k - (k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_l})}{n_k} \\
&= 1 - \varepsilon \frac{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_l}}{n_k}.
\end{aligned}$$

However

$$\frac{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_l}}{n_k} \geqslant \frac{|D|}{n_k} > \delta$$

(for large enough n_k), so

$$\frac{b_1 + b_2 + \cdots + b_{n_k}}{n_k} \leqslant 1 - \varepsilon \cdot \delta < 1$$

for large enough n_k.

2. There is an $\alpha_{l+1} \in D$ with $\alpha_{l+1} > \alpha_l + k_{\alpha_l} - 1$ (let α_{l+1} be the minimum of such elements). From the choice of α_l we have $n = \alpha_{l+1} + k_{\alpha_{l+1}} - 1 > n_k$. Then

$$\begin{aligned}\frac{b_1 + b_2 + \cdots + b_n}{n} &\leqslant (1-\varepsilon)\frac{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_{l+1}}}{n} + \frac{n - (k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_{l+1}})}{n} \\ &= 1 - \varepsilon\frac{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_{l+1}}}{n}.\end{aligned}$$

Recall that if $\kappa, \lambda > 0$ and $\vartheta > 0$, then $\frac{\kappa}{\lambda} \leqslant \frac{\kappa+\vartheta}{\lambda+\vartheta} \Leftrightarrow \frac{\kappa}{\lambda} \leqslant 1$, so we have

$$1 \geqslant \frac{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_{l+1}}}{n} \geqslant \frac{k_{\alpha_1} + k_{\alpha_2} + \cdots + k_{\alpha_l} + |\{m \in D\colon\ m > \alpha_l + k_{\alpha_l} - 1\}|}{n_k} \geqslant \frac{|D|}{n_k} > \delta$$

for large enough n_k. Finally

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \leqslant 1 - \varepsilon \cdot \delta$$

where $n > n_k$, n_k large enough.

In any case there is arbitrarily large $n \in \mathbb{N}$ such that

$$\frac{b_1 + b_2 + \cdots + b_n}{n} \leqslant 1 - \varepsilon \cdot \delta$$

which contradicts $d(M) = 1$. We conclude that $d(\complement A) = 0$ and thus $d(A) = 1$. □

Theorem 2. *Let $M \subseteq \mathbb{N}$ with $d(M) = 1$ and let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. Suppose there is a subsequence (k_n) of the positive integers with $f_{k_n}^\alpha / M \to_w (x, x, \ldots) \in \ell^\infty(M)$, $x \in \mathbb{R}$. Then there is an $N \subseteq M$ with $d(N) = 1$ such that α N-almost converges to x.*

If in particular $(\alpha_n)_{n\in\mathbb{N}}$ is weakly M-almost convergent to x, there is a subset N of M with $d(N) = 1$ such that $(\alpha_n)_{n\in\mathbb{N}}$ N-almost converges to x.

Proof. Since the value $x \in \mathbb{R}$ is of no importance, we prove the theorem for $x = 0$. For $n \in \mathbb{N}$ we denote

$$M_n = \left\{ m \in M\colon\ \frac{\chi_M(m) + \chi_M(m+1) + \cdots + \chi_M(m+k-1)}{k} > 1 - \frac{1}{n}\ \forall k \in \mathbb{N} \right\}.$$

Clearly $M = M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ and by the previous lemma $d(M_n) = 1$ $\forall n \in \mathbb{N}$. Similarly for $n, r \in \mathbb{N}$ we denote

$$M_n(r) = \left\{ m \in M_n\colon\ \frac{\chi_{M_n}(m) + \chi_{M_n}(m+1) + \cdots + \chi_{M_n}(m+k-1)}{k} > 1 - \frac{1}{r}\ \forall k \in \mathbb{N} \right\}$$

and again ($M_n(r) \subseteq M_n$ with) $d(M_n(r)) = 1$ $\forall n, r \in \mathbb{N}$.

We now construct an $N \subseteq M$ inductively as follows:

N contains all elements of M less than or equal to $m(2) = \min M_2(2)$.
N contains the elements of M_2 from $m(2)$ to $m(3) = \min(M_3(3) \setminus [1, m(2)])$.

In the nth step we put in N all elements of M_n from $m(n)$ to $m(n+1) = \min(M_{n+1}(n+1) \setminus [1, m(n)])$. It is clear that ($m(n) \in M_n$ and) $(m(n))$ is strictly increasing.

By the way N is constructed (note that $m(n) \in M_n(n)$, $n \in \mathbb{N}$) one can check that when $k \in \mathbb{N}$, with $k \geqslant m(n)$ then

$$\frac{\sum_{j=m(n)}^{k} \chi_N(j)}{k - m(n) + 1} > 1 - \frac{1}{n}. \tag{1}$$

We prove that $d_N^+(\alpha) = d_N^-(\alpha) = 0$.

If $d_N^+(\alpha) > \varepsilon > 0$ then consider a sequence (l_n) in N and a subsequence (p_n) of the positive integers such that

$$\frac{\alpha_{l_n} + \cdots + \alpha_{l_n+p_n-1}}{p_n} \geqslant \varepsilon \quad \forall n \in \mathbb{N}. \tag{2}$$

We may suppose that (l_n) is strictly increasing (see the definition of d_N^+ in the Preliminaries, also [9, Remarks 2(1)]).

Then (choosing a subsequence of (l_n) if necessary) we can assume that the terms l_n belong to different blocks of the form $[m(j), m(j+1)) \cap N$ of those constituting N and we can also assume that $p_n > n^2$ $\forall n \in \mathbb{N}$.

Claim. *Take an $n_0 \in \mathbb{N}$ so that for $n \geqslant n_0$ we have $\frac{2}{n} < \frac{\varepsilon}{2}$. Then for $n \geqslant n_0$ there is $x_n \in [l_n, l_n + p_n - 1] \cap M$ such that*

$$\frac{\alpha_{x_n} + \alpha_{x_n+1} + \cdots + \alpha_{x_n+k-1}}{k} \geqslant \frac{\varepsilon}{2} \quad \textit{for } k = 1, 2, \ldots, n.$$

Proof. If the claim does not hold, then for some $n \geqslant n_0$ and for each $m \in [l_n, l_n + p_n - 1] \cap M$ there is a $k(m)$, $1 \leqslant k(m) \leqslant n$, such that

$$\frac{\alpha_m + \alpha_{m+1} + \cdots + \alpha_{m+k(m)-1}}{k(m)} < \frac{\varepsilon}{2}.$$

Let now

$$\begin{aligned} &t_1 = l_n, \\ &t_2 = \min\bigl([l_n, l_n + p_n - 1] \cap M \setminus \bigl[t_1, t_1 + k(t_1) - 1\bigr]\bigr), \\ &\vdots \end{aligned}$$

the last term is

$$t_v = \min\left([l_n, l_n + p_n - 1] \cap M \setminus \bigcup_{i=1}^{v-1} \bigl[t_i, t_i + k(t_i) - 1\bigr]\right)$$

the *maximum* of those satisfying $t_v + k(t_v) - 1 \leqslant l_n + p_n - 1$.

Assume without loss of generality that $\|\alpha\|_\infty \leqslant 1$. We now have

$$\frac{\alpha_{l_n} + \cdots + \alpha_{l_n+p_n-1}}{p_n} \leqslant \frac{\varepsilon}{2} \frac{k(t_1) + \cdots + k(t_v)}{p_n} + \frac{|\complement M \cap [l_n, l_n + p_n - 1]|}{p_n} + \frac{n-1}{p_n}$$

(the third fraction standing because there may be a $t_{v+1} > t_v$, $t_{v+1} \in [l_n, l_n + p_n - 1] \cap M$ with $t_{v+1} + k(t_{v+1}) - 1 > l_n + p_n - 1$)

$$< \frac{\varepsilon}{2} + \frac{1}{n} + \frac{n}{n^2}$$

(because of the choice of (l_n) we have that $l_n \in M_n$, so the second fraction is less than $\frac{1}{n}$ and the rest are due to our assumptions)

$$= \frac{\varepsilon}{2} + \frac{2}{n} < \varepsilon$$

(from the choice of n).

We have a contradiction (because of (2)), so the claim holds. □

Now the sequence $(x_n) \subseteq M$, $n \geqslant n_0$ we obtained satisfies

$$f_k^\alpha(x_n) = \frac{\alpha_{x_n} + \alpha_{x_n+1} + \cdots + \alpha_{x_n+k-1}}{k} \geqslant \frac{\varepsilon}{2} \quad \text{for } n \geqslant n_0,\ k = 1, 2, \ldots, n, \tag{$\circledast$}$$

and there is a *contradiction* from the fact that there is a subsequence (k_n) of $\mathbb{N}$ such that $f_{k_n}^{\alpha}/M \to_w 0$. (Since α is a bounded sequence, by a diagonal process there is a subsequence (x_{j_n}) of (x_n) such that the $\lim_r f_{k_n}^{\alpha}(x_{j_r})$ exists $\forall n \in \mathbb{N}$. The double limit condition yields that $\lim_n \lim_r f_{k_n}^{\alpha}(x_{j_r}) = 0$, so for $\varepsilon > 0$ there is an $n_1 \in \mathbb{N}$ so that for $n \geqslant n_1$ we have $|f_{k_n}^{\alpha}(x_{j_r})| < \frac{\varepsilon}{2}$ for all except for finite $r \in \mathbb{N}$, which cannot happen due to $(\ast)$.)

Therefore $d_N^{+}(\alpha) \leqslant 0$ and similarly can be proved that $d_N^{-}(\alpha) \geqslant 0$, which imply that α is N-almost convergent to 0 (see [9, Theorem 1]).

Finally we will check that $d(N) = 1$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$ with $(1 - \frac{1}{n})^2 > 1 - \varepsilon$. Let also $k \in \mathbb{N}$: $\frac{k-m(n)+1}{k} > 1 - \frac{1}{n}$. Then (because of (1)) we have

$$\frac{\sum_{i=1}^{k} \chi_N(i)}{k} \geqslant \frac{\sum_{i=m(n)}^{k} \chi_N(i)}{k} > \frac{(1 - \frac{1}{n})(k - m(n) + 1)}{k} > \left(1 - \frac{1}{n}\right)^2 > 1 - \varepsilon$$

and the conclusion follows. □

Remarks 5. (1) Theorem 2 generalizes the Mean Ergodic Theorem in the special case of the shift operator $T : \ell^{\infty}(\mathbb{N}) \to \ell^{\infty}(\mathbb{N})$. Indeed, if we put in Theorem 2, $M = \mathbb{N}$ then (as we have already mentioned before Example 3) the Mean Ergodic Theorem gives that $N = \mathbb{N}$ and that α is almost convergent.

(2) It should be clear that the conclusion of Theorem 2 also holds true under the (stronger) assumption that the set $\{T^n\alpha / M\colon n \geqslant 1\}$ (for $d(M) = 1$) is a weakly relatively compact subset of $\ell^{\infty}(M)$. We recall in this connection that a sequence $\alpha \in \ell^{\infty}(\mathbb{N})$ is said to be *weakly almost periodic*, iff the set $\{T^n\alpha\colon n \geqslant 1\}$ is weakly relatively compact in $\ell^{\infty}(\mathbb{N})$ (see [2, pp. 316–317]).

(3) We note that the set N in the previous theorem on which the convergence of (f_n^{α}) is uniform does *not* depend on the specific sequence α, it only depends on the original set M.

From the previous theorem and [1, Theorem 13] (see also [9, Theorems 4, 5]) one can easily obtain the following:

Corollary 2. *Let K be a compact subset of $\mathbb{R}$ with at least two points. Let $X = K^{\mathbb{N}}$ and let μ be a strictly positive, regular Borel probability measure on K. Then for μ_{∞}-almost all points $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in X$ there is no $M \subseteq \mathbb{N}$ with $d(M) = 1$ so that α is weakly M-almost convergent.*

Remark 6. P.C. Baayen and G. Helmberg in [1] defined the notion "almost well distribution-M" (M an infinite subset of $\mathbb{N}$) for a sequence (x_n) in a compact, Hausdorff topological space X with respect to a regular Borel probability measure μ on X, requiring that for any continuous function $f : X \to \mathbb{R}$ the sequence $(f(x_n))_{n\in\mathbb{N}}$ M-almost converges to $\int_X f\,d\mu$ (see also [9, Definitions 1 and 6])

We will say that (x_n) is μ-weakly almost well distributed-M in X, if for any continuous function $f : X \to \mathbb{R}$ the sequence $(f(x_n))_{n\in\mathbb{N}}$ weakly M-almost converges to $\int_X f\,d\mu$.

One can obtain examples distinguishing the two notions from one another and from the classical uniform distribution. In case when (x_n) is weakly almost well distributed-M with $d(M) = 1$ then there is an $N \subseteq M$ with $d(N) = 1$ such that (x_n) is almost well distributed-N (from Theorem 2, using Remarks 5(3)).

Let M be an infinite subset of $\mathbb{N}$. We denote $\mathcal{BL}_M^w = \{L \in \mathcal{BL}\colon\ \alpha \text{ is weakly } M\text{-almost convergent to } x \Rightarrow L(\alpha) = x\}$ (the set of Banach limits which preserve the weak M-almost convergence). It is easy to check that this is a convex and weak-$\ast$ closed subset of the unit ball of $\mathcal{M}(\beta\mathbb{N})$. Following M. Jerison in [6] (see also [9, Section 2]) we define the maximal value of Banach limits which preserve the weak M-almost convergence.

Definition 3. Let $\tau_M^w : \ell^{\infty}(\mathbb{N}) \to \mathbb{R}$ be the functional with

$$\tau_M^w(\alpha) = \sup_{L\in\mathcal{BL}_M^w} L(\alpha).$$

The functional τ_M^w is sublinear (see Remarks 3(1)) and (obviously) when $f_n^{\alpha} / M \to_w (x, x, \ldots)$ then $\tau_M^w(\alpha) = x$. One can easily verify (in exactly the same way the corresponding argument about the maximal value of all Banach limits τ is proved, see [6, p. 87]) that τ_M^w is the *maximum* sublinear functional satisfying:

(♦) If $\varphi : \ell^{\infty}(\mathbb{N}) \to \mathbb{R}$ is a linear function with $\varphi(\alpha) \leqslant \tau_M^w(\alpha)$ $\forall \alpha \in \ell^{\infty}(\mathbb{N})$, then $\varphi \in \mathcal{BL}_M^w$.

Remark 7. If $L : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ is a linear functional with $L(\alpha) \leqslant w_M^+(\alpha)$ $\forall \alpha \in \ell^\infty(\mathbb{N})$ then $L(\alpha) = -L(-\alpha) \geqslant -w_M^+(-\alpha) = w_M^-(\alpha)$, hence $L \in \mathcal{BL}_M^w$. So by (♦) we have $w_M^+(\alpha) \leqslant \tau_M^w(\alpha)$ $\forall \alpha \in \ell^\infty(\mathbb{N})$.

In case of the functional τ (the maximal value of all Banach limits), one can prove that $d_{\mathbb{N}}^+ \equiv \tau$ (see [9,14]). In our case though we have the following:

Example 5. We give an example of a subset M of $\mathbb{N}$ and a sequence $\alpha \in \ell^\infty(\mathbb{N})$ such that $w_M^+(\alpha) < \tau_M^w(\alpha)$.

Let $M = \{2^{2n}\colon n \in \mathbb{N}\}$ and α be the characteristic function of the set $A = \bigcup_{n=1}^\infty [2^{2n+1}+1, 2^{2n+2}]$. One can check that $d_M^+(\alpha) = \frac{2}{3}$ and hence $w_M^+(\alpha) \leqslant d_M^+(\alpha) = \frac{2}{3}$ (see Proposition 2).

Consider the positive and strongly regular summation method (see [7, p. 216], [10]) $(A_n)_{n\in\mathbb{N}}$ with

$$A_n = \frac{1}{2^{2n+1}} \sum_{t\in[2^{2n+1}+1, 2^{2n+2}]} \delta_t, \quad n \in \mathbb{N}$$

(where δ_t denotes the Dirac measure on $\{t\}$). Let L be a weak-$*$ limit point of (A_n) in $\mathcal{M}(\beta\mathbb{N})$. Then L is a Banach limit (see [10, Theorem 2.1]) and obviously $L(\alpha) = 1$, since $A_n(\alpha) = 1$ $\forall n \in \mathbb{N}$.

We observe that if a sequence $b \in \ell^\infty(\mathbb{N})$ is Cesaro summable to $x \in \mathbb{R}$ then $\lim_n A_n b = x$ since:

$$\frac{b_{2^{2n+1}+1} + \cdots + b_{2^{2n+2}}}{2^{2n+1}} = 2 \cdot \frac{b_1 + \cdots + b_{2^{2n+2}}}{2^{2n+2}} - \frac{b_1 + \cdots + b_{2^{2n+1}}}{2^{2n+1}} \to 2x - x = x.$$

Hence the same is valid for weak M-almost convergence, which implies ($L \in \mathcal{BL}_M^w$, so) $\tau_M^w(\alpha) = 1 > w_M^+(\alpha)$.

Question. Is there an expression of the value $\tau_M^w(\alpha)$ containing only the terms of the sequence α? (For such expressions of the functional τ, see [9,14].)

We give two examples referring to the case of (f_n^α) being weakly Cauchy or not. We note in particular that when (f_n^α) is weakly Cauchy, then it converges pointwise on $\mathbb{N}$, so α is Cesaro summable (see the Preliminaries for the related concepts).

Examples 6. (1) There is an $\alpha = \chi_A$ such that (f_n^α) is weakly Cauchy, while $A \notin \mathcal{W}_M$ for any infinite $M \subseteq \mathbb{N}$.

Let $A = \bigcup_{n=1}^\infty [n^2, n^2+n]$. The sequence (f_n^α) is weakly Cauchy, since $d(A) = \frac{1}{2}$ and $f_n^\alpha(u) \to 0$ or 1 (see Lemma 1) $\forall u \in \beta\mathbb{N} \setminus \mathbb{N}$. Moreover $A \notin \mathcal{W}_M$ for any infinite $M \subseteq \mathbb{N}$ (see Example 2).

(2) There is a sequence $\alpha \in \ell^\infty(\mathbb{N})$ which is M-almost convergent for an $M \subseteq \mathbb{N}$ of density 1, while (f_n^α) is not weakly Cauchy.

Let $M = \bigcup_{n=1}^\infty [n^4, n^4 + 4n^3]$. It is easy to check that $d(M) = 1$. Consider a zero-one sequence $b = (b_n)$ such that the sequence $(T^k b)_{k\in\mathbb{N}}$ is dense in $\{0,1\}^{\mathbb{N}}$ (for instance (b_n) can be the sequence of digits of a normal number in base 2, see [7, p. 69] and also [9]). The sequence α is constructed as follows: In the last n points of each interval $[n^4, (n+1)^4)$ we put the first n points of b. The sequence α is the characteristic function of the union of these intervals for every $n \in \mathbb{N}$. One can check that α M-almost converges to 0 (see also [9, Example 2]). We now show that the sequence $(T^n\alpha)_{n=0}^\infty$ (where $T^0\alpha = \alpha$) is equivalent in the supremum norm to the usual basis (e_n) of ℓ^1, i.e. that there is a $\delta > 0$ so that $\forall n \in \mathbb{N}$ and for every choice of real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ the relation

$$\left\| \sum_{i=0}^{n-1} \lambda_{i+1} T^i \alpha \right\| \geqslant \delta \cdot \left\| \sum_{i=1}^{n} \lambda_i e_i \right\|_1$$

holds ($\|\cdot\|_1$ is the usual norm of ℓ^1).

Let $A_j = \{p \in \mathbb{N}\colon\ T^j\alpha(p) = 1\}$ and $B_j = \{p \in \mathbb{N}\colon\ T^j\alpha(p) = 0\}$, $j = 0, 1, 2, \dots$. It suffices to show that the sequence $(A_j, B_j)_{j=0}^\infty$ is independent (see also [9, proof of Theorem 3]).

Set $\varepsilon_j A_j = A_j$ when $\varepsilon_j = 1$ and $\varepsilon_j A_j = B_j$ when $\varepsilon_j = 0$. Consider any fixed choice of ε_j, $j = 0, 1, \dots, m$. Then there is a $p \in \mathbb{N}$ with $b_{p+j} = \varepsilon_j$, $j = 0, 1, 2, \dots, m$, so pick the first block of the sequence α with length $> p + m$. Suppose this block starts in $n_0 + 1$ coordinate. Then $\alpha_{n_0+p+j} = b_{p+j} = \varepsilon_j$, $j = 0, 1, 2, \dots, m \Leftrightarrow T^j\alpha(n_0 + p) = \varepsilon_j$,

$j = 0, 1, 2, \ldots, m \Leftrightarrow n_0 + p \in \bigcap_{j=0}^{m} \varepsilon_j A_j \neq \emptyset$ and a well-known result of Rosenthal for ℓ^1-embedding [3, Proposition 3, p. 207] gives the result. Hence the sequence (f_n^α) is not weakly Cauchy, since the sequence $(\sum_{i=1}^{n} \frac{1}{n} e_i)_{n\in\mathbb{N}}$ itself is not.

We close this section with the following characterization of almost convergence.

Theorem 3. *Let $\alpha \in \ell^\infty(\mathbb{N})$ such that (f_n^α) admits a weakly Cauchy subsequence $(f_{k_n}^\alpha)$ converging pointwise on $\beta\mathbb{N}$ to f^α (and let $f^\alpha(n) = x$ $\forall n \in \mathbb{N}$). Then the following are equivalent*:

1. *α is almost convergent to the value x.*
2. *For every infinite subset M of $\mathbb{N}$ there is a $u \in \beta\mathbb{N} \setminus \mathbb{N}$ with $M \in u$, such that u is a continuity point of the function f^α (it suffices that $f^\alpha(u) = x$).*

Proof. We only need to prove that (2) $\Rightarrow$ (1) since the other implication is obvious. Without loss of generality let $\alpha_n \in [0, 1]$ $\forall n \in \mathbb{N}$ (otherwise we can find a constant c such that $b_n = \alpha_n + c \geqslant 0$ $\forall n \in \mathbb{N}$ and divide by $\|(b_n)\|_\infty$).

Suppose that the sequence (f_n^α) does not converge uniformly. Then either $d_{\mathbb{N}}^+(\alpha) > x$ or $d_{\mathbb{N}}^-(\alpha) < x$. So let $d_{\mathbb{N}}^+(\alpha) > x + \varepsilon > x$ for some $\varepsilon > 0$. N. Hindman has given the following expression (under our assumptions) of the functional $d_{\mathbb{N}}^+$ (see [5, Theorem 5]):

$$d_{\mathbb{N}}^+(\alpha) = \sup\Bigg\{ y \geqslant 0\colon \text{ there is a subsequence } (t_n) \text{ of } \mathbb{N} \text{ such that for every } n \in \mathbb{N} \text{ and every } k \leqslant n \\ \text{we have } \frac{\alpha_{t_n} + \alpha_{t_n+1} + \cdots + \alpha_{t_n+k-1}}{k} \geqslant y \Bigg\}.$$

Let (t_n) be a (strictly increasing) subsequence of $\mathbb{N}$ such that

$$(*) \qquad f_k^\alpha(t_n) = \frac{\alpha_{t_n} + \alpha_{t_n+1} + \cdots + \alpha_{t_n+k-1}}{k} \geqslant x + \varepsilon$$

for every $n \in \mathbb{N}$ and every $k \leqslant n$. Set $M = \{t_n\colon n \in \mathbb{N}\}$ and consider $u \in \overline{M} \setminus M$ a continuity point of f^α (hence $f^\alpha(u) = x$). Since $f_{k_n}^\alpha(u) \to f^\alpha(u) = x$, there is an $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ we have

$$f_{k_n}^\alpha(u) \in \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2} \right).$$

Then, for any $n \geqslant n_0$ there is a $B_n \subseteq \mathbb{N}$ with $B_n \in u$ such that

$$(**) \qquad f_{k_n}^\alpha(B_n) \subseteq \left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2} \right).$$

Let now $n_1 \geqslant n_0$ and $l_n \geqslant k_{n_1}$ with $t_{l_n} \in B_{n_1}$ (there is one, since $B_{n_1} \cap M \in u$ is infinite). Then from relation $(*)$ we have

$$f_{k_{n_1}}^\alpha(t_{l_n}) \geqslant x + \varepsilon$$

and because of $(**)$ we have a contradiction. The case $d_{\mathbb{N}}^-(\alpha) < x$ is analogous. $\square$

We close the paper with the following

Question. Let $\alpha \in \ell^\infty(\mathbb{N})$ and $M \subseteq \mathbb{N}$ infinite. Assume that the set $\{T^n\alpha / M\colon n \geqslant 1\}$ is norm (respectively weakly) relatively compact in the Banach space $\ell^\infty(M)$. Is then the set $\{T^n\alpha\colon n \geqslant 1\}$ itself norm (respectively weakly) relatively compact in $\ell^\infty(\mathbb{N})$? Is in particular the sequence α almost convergent? (See Remarks 5(2) and [2, pp. 315–318].)

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